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## Rings which are sums of two subrings

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### Abstract

The main results concern radicals and polynomial identities of rings which are sums of two subrings. It is proved that a ring which is a sum of a nil subring of bounded index and a ring satisfying a polynomial identity also satisfies a polynomial identity. Rings which are sums of two Jacobson radical subrings are classified. Several open questions are answered. © 1998 Elsevier Science B.V. All rights reserved.

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### 0. Introduction

Suppose that an associative ring  $R$  has the form  $R = R_1 + R_2$ , where  $R_1$  and  $R_2$  are subrings of  $R$  (we keep this notation throughout the paper). One can ask what properties of  $R$  can be derived if  $R_1$  and  $R_2$  satisfy certain conditions. This, natural and interesting on its own question, was also inspired by many earlier studies of similar problems in other sections of algebra (cf. [2]). It was for the first time (probably) posed in [22, 23] and then studied in many papers [6, 9–12]. However, the most inspired was Kegel's paper [10] in which he proved that if both  $R_1$  and  $R_2$  are nilpotent, then so is  $R$ . There were also many independent works which in fact concerned specific aspects of the problem. For instance some papers (cf. [8]) of Flanigan concerned the problem of a description of rings  $R = R_1 + R_2$ , with  $R_1$  and  $R_2$  isomorphic to given rings, under the additional assumption that  $R_1 \triangleleft R$ . Polin [16] proved that a ring  $P$  is projective in the category of all rings if and only if there is a free ring  $F$  containing a subring  $S$  isomorphic to  $P$  and an ideal  $I$  such that  $F$  is the direct sum of  $S$  and  $I$  as additive groups. In [15] Klein proved that if  $R_1$  and  $R_2$  are left (right) nil ideals of  $R$  of bounded index, then  $R$  is a nil ring of bounded index.

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One can ask why do we restrict ourselves to the case of two subrings only. The question was already raised by Kegel in [10]. More precisely, he asked whether a ring which is a sum of three nilpotent subrings must be nilpotent itself. In [4] Bokut’ proved that every algebra over a field can be embedded into a simple algebra which is a sum of three nilpotent subalgebras. This result shows that the cases of two and more subrings are totally different. Let us note however that some results (cf. [13]) can be extended from two to more subrings, if additional relations among the subrings are assumed.

The main studies of rings which are sums of two subrings concerned radicals and, recently, polynomial identities. We also concentrate on these topics. In particular, we characterize fields which are sums of two Jacobson radical subrings (Section 2), prove that a ring which is a sum of a nil ring of bounded index and a PI ring is also PI (Section 3) and prove that if there exists a simple ring which is a sum of a nil ring and a specific nil ring, then there also exists a simple nil ring (Section 4). Moreover, we comment on several results of [11] and answer questions mentioned there.

All rings considered in this paper are associative but are not assumed to have an identity. To denote that  $I$  is an ideal (left ideal, right ideal) of a ring  $A$ , we write  $I \triangleleft A$  ( $I <_l A$ ,  $I <_r A$ ).

Instead of “a ring  $A$  has property  $S$ ” we write “ $A$  is an  $S$ -ring” or just “ $A \in S$ ”. The term “radical” means, depending on the context, “radical class” or “radical property”. The prime radical will be denoted by  $\beta$ .

**1. Annihilators and radicals**

Given a ring  $A$ , define the *left (right) hyperannihilator*  $l(A)$  ( $r(A)$ ) of  $A$  to be the union  $\bigcup_{\alpha \geq 0} l_\alpha(A)$  ( $\bigcup_{\alpha \geq 0} r_\alpha(A)$ ), where

$$l_0(A) = r_0(A) = 0$$

and for  $\alpha \geq 1$

$$l_\alpha(A) = \left\{ x \in A \mid xA \subseteq \bigcup_{\beta < \alpha} l_\beta(A) \right\},$$

$$r_\alpha(A) = \left\{ x \in A \mid Ax \subseteq \bigcup_{\beta < \alpha} r_\beta(A) \right\}.$$

The intersection  $N(A) = l(A) \cap r(A)$  is called the *hyperannihilator* of  $A$ .

The ring  $A$  is said to be *left (right) T-nilpotent* if  $A = l(A)$  ( $A = r(A)$ ). It is easy to see that a ring  $A$  is left (right)  $T$ -nilpotent if and only if every non-zero homomorphic image of  $A$  has non-zero left (right) annihilator. A ring whose all non-zero homomorphic images have non-zero left or right annihilators is called *M-nilpotent*.

**Proposition 1.1.** *For every prime ideal  $P$  of  $R$ ,  $r(R_1) \subseteq P$  or  $l(R_2) \subseteq P$ . In particular if  $R$  is a prime ring, then  $r(R_1) = 0$  or  $l(R_2) = 0$ .*

**Proof.** Suppose that  $\alpha$  and  $\beta$  are ordinals such that if  $\alpha' < \alpha$  or  $\beta' < \beta$ , then  $l_{\alpha'}(R_1)r_{\beta'}(R_2) \subseteq P$ . Then  $l_{\alpha}(R_1)Rr_{\beta}(R_2) = l_{\alpha}(R_1)R_1r_{\beta}(R_2) + l_{\alpha}(R_1)R_2r_{\beta}(R_2) \subseteq P$ . However, the ideal  $P$  is prime, so  $l_{\alpha}(R_1)r_{\beta}(R_2) \subseteq P$ . Now simple induction arguments give that  $l(R_1)r(R_2) \subseteq P$ . Hence  $l(R_1)Rr(R_2) = l(R_1)R_1r(R_2) + l(R_1)R_2r(R_2) \subseteq P$ , and since  $P$  is prime,  $l(R_1) \subseteq P$  or  $r(R_2) \subseteq P$ .  $\square$

Let us note that if  $R$  is the ring  $M_2(K)$  of  $2 \times 2$ -matrices over a field  $K$  and  $R_1 = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ ,  $R_2 = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ , then  $R = R_1 + R_2$  is a simple ring and both  $r(R_1)$  and  $r(R_2)$  are non-zero. Hence, in Proposition 1.1 the left hyperannihilator cannot be replaced by the right hyperannihilator.

Proposition 1.1 gives in particular the following corollary which was proved in [11] in a more complicated way.

**Corollary 1.2.**  $N(R_1)N(R_2) \subseteq \beta(R)$  and  $N(R_2)N(R_1) \subseteq \beta(R)$ .

**Remark 1.3.** To be precise, in [11] Kegel considered not  $\beta(R)$  but the sum of all solvably embedded ideals of  $R$ . However, one can easily check that they both coincide.

Kegel asked whether at least one of  $N(R_1)$  and  $N(R_2)$  is contained in  $\beta(R)$  (or at least in the locally nilpotent radical). The following easy example answers the question in the negative.

**Example 1.4.** Let  $R = M_2(K) \oplus M_2(K)$ , where  $K$  is a field,  $R_1 = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \oplus M_2(K)$  and  $R_2 = M_2(K) \oplus \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ . Clearly  $R = R_1 + R_2$ ,  $N(R_1) \neq 0 \neq N(R_2)$  but  $R$  is even Brown–McCoy semisimple.

The example also shows that there are no direct relations between radicals of  $R$  and those of  $R_1$  and  $R_2$  (at least for classical radicals). The situation improves when one of rings is radical. In [11] Kegel proved.

**Theorem 1.5.** *If  $R_2$  is locally nilpotent, then the subring of  $R$  generated by  $N(R_1)$  and  $R_2$  is locally nilpotent and  $N(R_1)$  is contained in the locally nilpotent radical of  $R$ .*

The idea of the proof would be hard to apply to other radicals. Another method was used in [6]. The proofs of Theorem 2 and Corollary 4 in [6] give

**Theorem 1.6.** *If  $R_2$  is Jacobson radical (locally nilpotent), then  $l(R_1) + r(R_1)$  is contained in the Jacobson (locally nilpotent) radical of  $R$ .*

**Remark 1.7.** From Theorem 1.6 it is easy to get that if  $R_2$  is Jacobson radical (locally nilpotent), then the subring of  $R$  generated by  $l(R_1) + r(R_1)$  and  $R_2$  is Jacobson radical (locally nilpotent). Indeed, note that if  $I$  is an ideal of  $R_1$ , then  $J = I + R_2I + IR_2 + R_2IR_2$  is the ideal of  $R$  generated by  $I$ . The subring of  $R$  generated by  $I$  and  $R_2$  is equal to  $J + R_2$ . Now it suffices to put  $I = l(R_1) + r(R_1)$  and apply Theorem 1.6.

Related is the following:

**Question** (Kelarev and McConnell [13]). Does there exist a ring which is not Jacobson radical but is a sum of a Jacobson radical subring  $E$  and an additive subgroup  $F$  such that  $F^2 \subseteq E$ ?

Recall that a radical  $S$  is called *hereditary* (*left hereditary*, *right hereditary*) if for every ring  $A \in S$  and each  $I \triangleleft A$  ( $I <_l A$ ,  $I <_r A$ ),  $I \in S$ . Hereditary radicals containing the prime radical  $\beta$  are called *supernilpotent*.

A radical  $S$  is called *left* (*right*) *strong* if for every ring  $A$ ,  $S(A)$  contains all left (right)  $S$ -ideals of  $A$ . It is known [21] that if a radical  $S$  containing  $\beta$  is left or right strong and left or right hereditary, then it is left and right strong as well as left and right hereditary. Radicals of that type are called *N-radicals*. Examples of *N-radicals* are the prime, locally nilpotent and Jacobson radicals. The problem whether the nil radical is an *N-radical* is still open; it is called Koethe's problem.

Theorem 1 in [14] gives

**Theorem 1.8.** *If  $R_1 \in \beta$  and  $R_2 \in S$ , where  $S$  is a right strong and supernilpotent radical, then  $l(R_1) \subseteq S(R)$ .*

It is bit surprising that, as an example in [14] shows, the assumption  $R_1 \in \beta$  in Theorem 1.8 cannot be dropped. The problem of finding a reasonable extension of Theorem 1.8 seems to be quite interesting.

As an immediate consequence of Theorem 1.8 one gets

**Corollary 1.9** (Kępczyk and Puczyłowski [14, Corollary 1]). *Suppose that  $S$  is a supernilpotent radical and  $R_2 \in S$ .*

*If  $R_1$  is a left (right) T-nilpotent ring and the radical  $S$  is right (left) strong, then  $R \in S$ .*

*If  $R_1$  is an M-nilpotent ring and the radical  $S$  is left and right strong, then  $R \in S$ .*

Corollary 1.9 applies in particular to the Jacobson, locally nilpotent and prime radicals. The corollary holds for the nil radical (even with  $R_1$  nilpotent of index 2) if and only if the Koethe's problem has a positive solution [6]. Now it is natural to ask whether one can get results similar to that of Corollary 1.9 with other assumptions on  $R_1$ . The following theorem is of such type.

**Theorem 1.10** (Kępczyk and Puczyłowski [14, Theorem 2]). *If  $R_1$  is a nil PI ring and  $R_2 \in S$ , where  $S$  is an  $N$ -radical, then  $R \in S$ .*

Now we shall prove

**Proposition 1.11.** *If  $R_1 \in \beta$  and  $R_2$  satisfies a multilinear identity*

$$\sum_{\pi \in S_{d+1}} \alpha_\pi x_{\pi(1)} \dots x_{\pi(d+1)} \equiv 0 \quad \text{with} \quad \sum_{\pi(d+1)=d+1} \alpha_\pi = 1,$$

then  $R_1 \subseteq \beta(R)$ .

**Proof.** We can assume that  $R$  is a prime ring and then we have to show that  $R_1 = 0$ . If  $R_1 \neq 0$ , then it contains a non-zero nilpotent ideal  $I$ . Let  $k$  be a natural number such that  $I^{k+1} = 0$  and  $I^k \neq 0$ . Put  $L = I + RI$  and  $P = R_1 + L$ . Observe that  $L <_l R$  and  $L \triangleleft P$ . Moreover, the modularity of the lattice of additive subgroups of  $R$  implies that  $P = R_1 + P \cap R_2$ . If  $P \cap R_2$  were nil, then being a PI ring, it would be  $\beta$ -radical. Hence, by Theorem 1.10 we would get that  $P \in \beta$  and consequently  $L \in \beta$ . This is impossible because  $L$  is a non-zero left ideal of  $R$  and  $R$  is prime. Thus,  $P \cap R_2$  is not nil.

Now,  $P \cap R_2 / L \cap R_2 = P \cap R_2 / L \cap P \cap R_2 \simeq (P \cap R_2 + L) / L \subseteq P / L$  and  $P / L$  is a homomorphic image of  $R_1$ . Hence,  $P \cap R_2 / L \cap R_2$  is a nil ring. Consequently  $L \cap R_2$  is not nil. Let  $t$  be a non-nilpotent element of  $L \cap R_2$ . Since  $L = I + RI$  and  $I^{k+1} = 0$ , we have  $tI^k = 0$ . Substituting in the multilinear identity satisfied by  $R_2$ ,  $x_1 = x_2 = \dots = x_d = t$ , we get that

$$t^d R_2 \subseteq t^{d-1} R_2 t + t^{d-2} R_2 t^2 + \dots + R_2 t^d.$$

Consequently  $t^d R_2 I^k = 0$  and  $t^d R I^k = t^d R_1 I^k + t^d R_2 I^k = 0$ . Therefore,  $R$  is not prime, a contradiction.  $\square$

Obviously Proposition 1.11 can be applied when  $R_2$  is a commutative ring. Thus, we have

**Corollary 1.12.** *If  $R_1 \in \beta$  and  $R_2$  is commutative, then  $R_1 \subseteq \beta(R)$ .*

Note that the assumption on the identity in Proposition 1.11 cannot be dropped, which one can see taking  $R = R_2$  the ring of  $n \times n$ -matrices,  $n \geq 2$ , over a field and  $R_1$  any non-zero nilpotent subring of  $R$ .

In [11] Kegel asked whether the subring of  $R$  generated by the locally nilpotent radicals of  $R_1$  and  $R_2$  must be locally nilpotent or at least nil. This question has an easy negative answer. Let  $R$  be the ring  $M_2(K)$  of  $2 \times 2$ -matrices over a field  $K$  and  $R_1 = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ ,  $R_2 = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ . Obviously,  $R_1$  is locally nilpotent (even  $R_1^2 = 0$ ), the locally nilpotent radical of  $R_2$  is equal to  $\beta(R_2) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$  and the subring generated by the radicals is equal to  $R$ . However, the problem (cf. [18]) is not so easy if one assumes that both  $R_1$  and  $R_2$  are nil (locally nilpotent, prime radical) and it stimulated several

studies in the area. In [12] Kelarev constructed an example of a nil semisimple ring which was a sum of two locally nilpotent subrings. Recently, Salwa [20] developed Kelarev’s ideas and found simpler examples of the sort. They in particular show (though it was not emphasized in the paper) that a Jacobson semisimple ring can be a sum of two subrings which are sums of nilpotent ideals. We raise the following:

**Question.** Can a ring which is a sum of two nil (locally nilpotent, prime radical) subrings contain a non-zero idempotent?

Some results related to this questions can be found in [7].

We conclude this section by showing that there are non-strict radicals  $S$  such that  $R \in S$  provided both  $R_1$  and  $R_2$  are in  $S$ . The question of whether it is true was suggested to us by B.J. Gardner.

Recall that a radical  $S$  is called *strict* if all  $S$ -subrings of each ring  $A$  are contained in  $S(A)$ .

**Theorem 1.13.** *The upper radical  $S$  determined by the group ring  $A = F_2[C_2]$  of a cyclic group  $C_2$  of order 2 over a field  $F_2$  of two elements is non-strict but if both  $R_1$  and  $R_2$  are  $S$ -radical, then  $R$  is  $S$ -radical.*

**Proof.** Note that  $A = P + I$ , where  $P$  is a subring of  $A$  isomorphic to  $F_2$  and  $I \triangleleft A$ ,  $I^2 = 0$ . Moreover,  $P$  and  $I$  are the only non-trivial subrings of  $A$ . A ring is  $S$ -radical if and only if it cannot be homomorphically mapped neither onto  $A$  nor  $I$ . Clearly  $P$  is  $S$ -radical and  $A$  is  $S$ -semisimple. Hence  $S$  is not strict. Now if  $R$  is not  $S$ -radical, then it can be homomorphically mapped onto  $A$  or  $I$ . Hence,  $A$  or  $I$  is a sum of homomorphic images of  $R_1$  and  $R_2$ . In both cases  $I$  is a homomorphic image of  $R_1$  or  $R_2$  and hence  $R_1$  or  $R_2$  is not  $S$ -radical.  $\square$

## 2. Fields which are sums of two Jacobson radical subrings

In [6] it was noted that the field of rational numbers is a sum of two Jacobson radical subrings. Here we characterize all fields of that type. We shall need the following lemma which is perhaps well known. We sketch its short proof for completeness.

**Lemma 2.1.** *Let  $M \subseteq L$  be subfields of an algebraic closure  $\bar{F}_p$  of a finite prime field  $F_p$ .*

- (i) *If  $(L : M) < \infty$ , then  $L = MK$ , where  $K = F_p(a)$  is a finite subfield of  $\bar{F}_p$ . Moreover  $(L : M) = (K : K \cap M)$ .*
- (ii) *For every natural number  $t$  dividing  $(L : M)$  there exists a unique subfield  $T$  of  $L$  such that  $M \subseteq T$  and  $(T : M) = t$ .*
- (iii) *If  $a, b \in \bar{F}_p$ ,  $(M(a) : M) = r$ ,  $(M(b) : M) = s$  and  $r, s$  are relatively prime, then  $(M(a + b) : M) = rs$ .*

**Proof.** (i) As  $K$  one can take for instance  $F_p$  extended by a basis of  $L$  over  $M$ . From the theory of finite fields it is known that  $K = F_p(a)$  for some  $a \in \bar{F}_p$  and  $K$  is a splitting field over  $F_p$  of a polynomial  $x^{p^n} - x$ . If  $f(x) \in M[x]$  is a minimal polynomial for  $a$  over  $M$ , then  $f(x)$  divides  $x^{p^n} - x$  in  $M[x]$ . This implies that all roots of  $f(x)$  belong to  $K$ . Consequently  $f(x) \in (K \cap M)[x]$ . Hence,  $(L : M) = (M(a) : M) = \deg f(x) = ((K \cap M)(a) : (K \cap M)) = (K : K \cap M)$ .

(ii) By (i),  $(L : M) = (K : K \cap M)$ , so  $t$  divides  $(K : K \cap M)$ . From the theory of finite fields it is known that there exists a unique subfield  $N$  of  $K$  such that  $K \cap M \subseteq N$  and  $(N : K \cap M) = t$ . Now, applying (i), it is not hard to check that  $T = MN$  satisfies the hypothesis of (ii).

(iii) Clearly,  $(M(a, b) : M) = rs$ , so  $(M(a + b) : M) = r's'$  for some divisors  $r'$  of  $r$  and  $s'$  of  $s$ . By (ii), there are subfields  $T'$  of  $M(a)$  and  $T''$  of  $M(b)$  containing  $M$  such that  $(T' : M) = r'$  and  $(T'' : M) = s'$ . Clearly,  $(T'T'' : M) = r's'$ , so the uniqueness in (ii) gives that  $M(a + b) = T'T''$ . Now it is enough to show that  $a, b \in T'T''$ . By (i),  $(T'T''(a) : T'T'') = (F_p(a) : T'T'' \cap F_p(a))$  and  $r = (M(a) : M) = (F_p(a) : M \cap F_p(a))$ . Clearly,  $(F_p(a) : T'T'' \cap F_p(a))$  divides  $(F_p(a) : M \cap F_p(a))$ , so  $(T'T''(a) : T'T'')$  divides  $r$ . Similarly  $(T'T''(b) : T'T'')$  divides  $s$ . But  $T'T''(a) = M(a + b, a) = M(a, b) = M(a + b, b) = T'T''(b)$ . Hence, since  $r$  and  $s$  are relatively prime,  $T'T''(a) = T'T''(b) = T'T''(b)$ .  $\square$

Now we are ready to prove

**Theorem 2.2.** *Given a field  $F$  the following are equivalent:*

- (a)  $F = A + B$  for some Jacobson radical rings  $A$  and  $B$ ;
- (b)  $F = A + B$  for some proper subrings  $A$  and  $B$  of  $F$ ;
- (c)  $\text{ch } F = 0$  or  $\text{tr}_{F_0} F > 0$ , where  $F_0$  is the prime subfield of  $F$ .

**Proof.** It is clear that (a) implies (b). Suppose now that (b) holds and (c) does not hold. Consequently,  $F$  is a subfield of an algebraic closure  $\bar{F}_p$  of a finite prime field  $F_p$ . Then  $A$  and  $B$  are locally finite and, being domains, they must be fields. To get a contradiction it suffices to show that  $F = A \cup B$ . Set  $M = A \cap B$  and take any  $f \in F$ . We have to show that  $f \in A \cup B$ . Clearly we can assume that  $(M(f) : M) = n > 1$  and for every  $x \in F$  with  $(M(x) : M) < n$ , we have  $x \in A \cup B$ . Take any  $y \in M(f)$ . If  $(M(y) : M) < n$ , then  $y \in M(f) \cap A + M(f) \cap B$ . Suppose now that  $(M(y) : M) = n$ , i.e.,  $M(y) = M(f)$ . Let  $y = a + b$  for some  $a \in A$  and  $b \in B$ . By Lemma 2.1 (ii), if  $(M(a) : M) = r$  and  $(M(b) : M) = s$ , then  $r$  and  $s$  are relatively prime. Hence by Lemma 2.1(iii),  $M(f) = M(y) = M(a + b) = M(a, b)$ , so again  $y \in M(f) \cap A + M(f) \cap B$ . Consequently,  $M(f) = M(f) \cap A + M(f) \cap B$ . This implies that if  $(M(f) \cap A : M) = k$  and  $(M(f) \cap B : M) = l$ , then  $(M(f) : M) = k + l - 1$ . However, both  $k$  and  $l$  divide  $(M(f) : M)$ , so  $k = 1$  or  $l = 1$ . Therefore  $f \in A \cup B$ . Thus (b) implies (c).

Suppose now (c). Then we can assume that  $F$  is an algebraic extension of the field  $Q$  of rational numbers or  $F$  is an algebraic extension of the field  $K = F_0(X)$  of rational functions over  $F_0$  in a non-empty set of indeterminates  $X$ . Let  $P$  denote the ring of

integers in the former case and the ring of polynomials over  $F_0$  in indeterminates from  $X$  in the latter. Let  $\bar{P}$  be the integral closure of  $P$  in  $F$ . Clearly  $F$  is equal to  $\bar{P}$  localized at  $S = P \setminus \{0\}$ , i.e.,  $F = \bar{P}S^{-1}$ . Let  $p, q$  be two distinct irreducible elements in  $P$  and let  $I$  and  $J$  be prime ideals in  $\bar{P}$  with  $I \cap P = pP$  and  $J \cap P = qP$ . Denote by  $A$  and  $B$  the Jacobson radicals of  $\bar{P}$  localized at  $\bar{P} \setminus I$  and  $\bar{P} \setminus J$ , respectively. Thus  $A = I(\bar{P} \setminus I)^{-1}$  and  $B = J(\bar{P} \setminus J)^{-1}$ . We claim that  $A + B = F$ . Since  $P = pP + qP \subseteq I + J$ , we have that  $\bar{P} = I + J$ . For every irreducible element  $s \in P$  and each natural number  $n$ ,  $s^{-n} \in A + B$ . This is clear if  $s \neq p, q$ . Now  $1 = p^{n+1}p_1 + qp_2$  for some  $p_1, p_2 \in P$ . Thus  $p^{-n} = pp_1 + qp_2p^{-n} \in A + B$ . Similarly  $q^{-n} \in A + B$ . Moreover, every element of  $S^{-1}$  is a sum of elements of the form  $ps^{-n}$ , where  $p \in P$  and  $s$  is an irreducible element of  $P$ . By the foregoing  $\bar{P}S^{-n} \subseteq (I + J)(A + B) \subseteq A + B$ . Hence, since  $F = \bar{P}S^{-1}$ ,  $F = A + B$ . The result follows.  $\square$

It seems that it would not be easy to describe the class of all rings which are sums of two Jacobson radical subrings.

A descending chain  $I_1 \supseteq I_2 \supseteq \dots$  of ideals of a ring  $A$  is said [11] to *absorb*  $A$  if for every natural number  $n$  one has

$$AI_n + I_nA \subseteq I_{n+1}.$$

In [11] Kegel asked whether the existence of properly descending absorbing chains of ideals of  $R_1$  and  $R_2$  is sufficient to force the ring  $R$  to be non-simple. The answer is negative. Indeed, let  $Q$  be the field of rational numbers and  $Q_1 = \{2n/(2m + 1) \mid n, m \text{ any integers}\}$ ,  $Q_2 = \{3k/l \mid k \text{ a integer and } l \text{ a integer relatively prime to } 3\}$ . From the proof of Theorem 2.2 (cf. also [6]) it follows that  $Q = Q_1 + Q_2$ . Clearly  $Q_1^n$  and  $Q_2^n$ ,  $n = 1, 2, \dots$ , are properly descending absorbing chains of ideals of  $Q_1$  and  $Q_2$ , respectively.

### 3. Polynomial identities

In [1] Bahturin and Giambruno proved that if  $R_1$  and  $R_2$  are commutative, then  $R$  is a PI ring satisfying the identity  $[x, y][z, t] \equiv 0$  (as usual  $[x, y] = xy - yx$ ). Beidar and Mikhalev [3] extended this result proving that if  $R_1$  satisfies the identity  $[x_1, y_1][x_2, y_2] \dots [x_n, y_n] \equiv 0$  and  $R_2$  satisfies the identity  $[x_1, y_1][x_2, y_2] \dots [x_m, y_m] \equiv 0$ , then  $R$  is a PI ring. They also asked

**Question.** Suppose that both  $R_1$  and  $R_2$  are PI rings. Is  $R$  a PI ring?

In this section we answer the question positively in the case when  $R_1$  is nil of bounded index. In the proof we follow some methods of [3,14].

Given a ring  $A$  we denote by  $W(A)$  the sum of nilpotent ideals of  $A$ . We follow the convention that  $A^0 \subseteq W(A)$  if and only if  $A$  is nilpotent.



**Lemma 3.1** (Kępczyk and Puczyłowski [14, Lemma 1]). *If  $a$  is an element of a ring  $A$  and for a natural number  $n$ ,  $A^n \subseteq W(A)$ , then  $(Aa)^{n-1} \subseteq W(Aa)$ .*

Now we shall prove

**Proposition 3.2.** *If  $R_1$  is a nil PI ring,  $R_2$  is a PI ring and  $R \neq 0$ , then  $R$  contains a non-zero left ideal which is a PI ring.*

**Proof.** For every nil PI ring  $A$ , there is (cf. [19, Theorem 1.6.36]) an integer  $n \geq 0$  such that  $A^n \subseteq W(A)$ .

We proceed by induction on  $n$  such that  $R_1^n \subseteq W(R_1)$ . Obviously, we can assume that  $R$  is semiprime. Suppose first that  $n=0$ , i.e.,  $R_1^m=0$  for a natural number  $m$ . For  $m=1$  the result is clear. Thus assume that  $m>1$ ,  $R_1 \neq 0$  and the result holds when the index of nilpotency of  $R_1$  is less than  $m$ . Let  $L=R_1+RR_1$ . It is easy to see that  $L$  is a left ideal of  $R$  and the right annihilator  $I$  of  $L$  in  $L$  contains  $R_1^{m-1}$ . Furthermore  $L=R_1+(L \cap R_2)$ . Now  $L/I=(R_1+I)/I+((L \cap R_2)+I)/I$ . Obviously  $(R_1+I)/I$  is nilpotent of index not greater than  $m-1$  and  $((L \cap R_2)+I)/I$  is a PI ring. Moreover, since  $R$  is semiprime,  $L/I$  is non-zero. Hence,  $L/I$  contains a non-zero left ideal  $K/I$  which is a PI ring. Obviously,  $K$  is also a PI ring and  $0 \neq LK \subseteq K$ , so  $LK$  is a non-zero left ideal of  $R$  which is a PI ring.

Suppose now that  $n>0$  and the result holds for smaller integers. Note that if  $I$  and  $J$  are ideals of  $R_1$  such that  $J I=0$ , then  $\bar{R}_1=R_1+IRJ$  is a subring of  $R$ ,  $IRJ$  is an ideal of  $\bar{R}_1$  such that  $(IRJ)^2=0$  and  $\bar{R}_1/IRJ$  is a homomorphic image of  $R_1$ . These imply that  $\bar{R}_1$  is a nil PI ring and  $(\bar{R}_1)^n \subseteq W(\bar{R}_1)$ . Suppose that  $0 \neq a \in \bar{R}_1 \cap R_2$  and put  $U=\{x \in Ra \mid Rax=0\}$ . Obviously  $U$  is an ideal of  $Ra$  and, since  $R$  is semiprime,  $U \neq Ra$ . By Lemma 3.1, we have  $(\bar{R}_1 a)^{n-1} \subseteq W(\bar{R}_1 a)$ . Hence, applying the induction assumption to  $Ra/U=(\bar{R}_1 a+U)/U+(R_2 a+U)/U$ , we get that  $Ra/U$  contains a non-zero left ideal  $T/U$  which is a PI ring. Clearly,  $RaT$  is a non-zero left ideal of  $R$  which is a PI ring. Hence, the result holds provided  $\bar{R}_1 \cap R_2 \neq 0$ . Thus let us assume that  $\bar{R}_1 \cap R_2=0$ . Then  $\bar{R}_1=\bar{R}_1 \cap (R_1+R_2)=R_1+(\bar{R}_1 \cap R_2)=R_1$ , which gives  $IRJ \subseteq R_1$ . In particular if  $K$  is an ideal of  $R_1$  and  $K^k=0$ , then for every  $1 \leq i \leq k-1$ ,  $A_i=K^{k-i}RK^i \subseteq R_1$ . The ring satisfies for a natural number  $d$  the identity

$$x_1 x_2 \dots x_d = \sum \alpha_\pi x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(d)},$$

where the sum ranges over all non-trivial permutations of the set  $\{1, 2, \dots, d\}$ . Hence, if  $k>d$ , then

$$(K^{k-1}R)^d K^d = A_1 A_2 \dots A_d \subseteq \sum A_{\pi(1)} A_{\pi(2)} \dots A_{\pi(d)} = 0,$$

so  $(K^{k-1}R)^{d+1}=0$ . Hence, since  $R$  is semiprime, for every nilpotent ideal  $K$  of  $R_1$ ,  $K^d=0$ . This easily implies that  $R_1$  is nilpotent, which contradicts the assumption that  $n>0$ . The proof is complete.  $\square$

**Theorem 3.3.** *If  $R_1$  is a nil ring of bounded index and  $R_2$  is a PI ring, then  $R$  is a PI ring.*

**Proof.** Let  $\mathcal{R}$  be the class of all rings  $R = R_1 + R_2$ , such that  $R_1$  is a nil ring of bounded index and  $R_2$  is a PI ring. Suppose that there is  $R \in \mathcal{R}$  which is not a PI ring. Let  $F = Z\langle x_1, x_2, \dots \rangle$  be the free ring in indeterminates  $x_1, x_2, \dots$  over the ring of integers  $Z$  and let  $M$  be the multiplicatively closed subset of  $F$  generated by all standard polynomials. For every  $0 \neq s \in M$  there is a homomorphism  $\varphi: F \rightarrow R$  such that  $\varphi(s) \neq 0$ . Hence there is a homomorphism  $\psi: F \rightarrow \prod R$  of  $F$  into a direct power  $\prod R$  such that  $0 \notin \psi(M)$ . Let  $T$  be an ideal of  $\prod R$  maximal with respect to  $T \cap \psi(M) = \emptyset$ . Clearly,  $(\prod R)/T$  is a prime ring and there is a homomorphism  $\bar{\psi}: F \rightarrow (\prod R)/T$  such that  $0 \notin \bar{\psi}(M)$ . Since every PI ring satisfies a power of a standard identity,  $(\prod R)/T$  is not a PI ring. Clearly,  $(\prod R)/T \in \mathcal{R}$ . Hence, we can assume that  $R$  is a prime ring. If  $k = \text{char}(R)$ , then the free  $Z/kZ$ -algebra  $K = (Z/kZ)\langle x_1, x_2, \dots \rangle$  can be embedded into a direct power  $\prod R$ . Let  $I$  be an ideal of  $\prod R$  maximal with respect to  $K \cap I = 0$ . Clearly,  $\bar{R} = (\prod R)/I \in \mathcal{R}$  and  $\bar{R}$  is prime. Moreover, we can assume that  $K \subseteq \bar{R}$  and for every non-zero ideal  $J$  of  $\bar{R}$ , we have  $J \cap K \neq 0$ . By Proposition 3.2,  $\bar{R}$  contains a non-zero left ideal which is a PI ring. Hence,  $\bar{R}$  is a GPI ring and by [19, Theorem 7.6.7], the Martindale's central closure  $Q$  of  $\bar{R}$  is a strongly primitive ring with a non-zero socle  $S$ . Now  $S \cap \bar{R}$  is a non-zero ideal of  $\bar{R}$ . Hence there is  $0 \neq f \in S \cap K$ . Take an idempotent  $e$  in  $S$  such that  $fe = f$ . By [19, Proposition 7.5.17(i)],  $Qe$  is a PI ring. Consequently  $Kf = Kfe \subseteq Qe$  is a PI ring, a contradiction.  $\square$

#### 4. Simple rings

In [11] Kegel asked whether the ring  $R$  is non-simple provided the locally nilpotent radicals of both  $R_1$  and  $R_2$  are non-zero. As we have already noted in Section 1 the ring  $R$  of  $2 \times 2$ -matrices over a field is a sum of a subring  $R_1$  such that  $R_1^2 = 0$  and a subring  $R_2$  with non-zero  $\beta$ -radical. However by Corollary 1.9 and Theorem 1.10, if  $R_2$  is locally nilpotent and  $R_1$  is  $M$ -nilpotent or nil PI, then  $R$  is locally nilpotent. No locally nilpotent ring can be simple. Thus in these cases  $R$  cannot be simple. As we have mentioned at the end of Section 1 there are Jacobson semisimple rings which are sums of two locally nilpotent subrings. These suggest

**Question.** Does there exist a simple ring which is a sum of two locally nilpotent ( $\beta$ -radical) subrings?

The rings of the mentioned examples of Kelarev and Salwa are non-simple because they are graded by the semigroup of natural numbers.

Let us recall that the question of whether there exists a simple nil ring is still open. Perhaps there is a better chance to construct a simple ring which is a sum of two nil subrings. We conclude this paper with some remarks which show in particular that in

some cases to construct such a simple ring is as hard as to construct a simple nil ring itself.

**Proposition 4.1.** *Let  $S \supseteq \beta$  be a right hereditary radical,  $\mathcal{A} = \{P \mid P \text{ contains no non-zero left } S\text{-ideal}\}$  and let  $T \triangleleft L <_1 A$ .*

- (i) *If  $I$  is an ideal of  $A$  maximal with respect to  $I \cap L \subseteq T$  and  $L/T \in \mathcal{A}$ , then  $A/I \in \mathcal{A}$ .*
- (ii) *If  $A \in \mathcal{A}$ , then  $L/S(L) \in \mathcal{A}$ .*

**Proof.** (i) Obviously, we can assume that  $I = 0$  and then we have to show that  $L/T \in \mathcal{A}$  implies  $A \in \mathcal{A}$ .

Suppose that  $K <_1 A$  and  $K \in S$ . Take any  $l \in L$ . Clearly,  $lK <_r K$ , so the hereditariness of  $S$  gives  $lK \in S$ . It is not hard to check (cf. [17, Lemma 2]), that the map  $rl + \beta(A) \rightarrow lr + \beta(lA)$  is a ring isomorphism of  $Al/\beta(A)$  onto  $lA/\beta(lA)$ . In particular  $(Kl + \beta(A))/\beta(A) \simeq (lK + \beta(lA))/\beta(lA)$ . Hence, since  $\beta \subseteq S$ , we have  $Kl \in S$ . However,  $Kl <_1 L$  and  $L/T \in \mathcal{A}$ , so  $Kl \subseteq T$ . Consequently,  $KL \subseteq T$  and  $(K + KA)L \subseteq T$ . Now  $J = K + KA \triangleleft A$  and  $(J \cap L)^2 \subseteq JL \subseteq T$ , which together with  $\beta \subseteq S$  and  $S(L/T) = 0$ , give  $J \cap L \subseteq T$ . Hence (since we assumed that  $I = 0$ )  $J = 0$ . Thus  $K = 0$  and we are done.

(ii) Suppose that  $S(L) \subseteq K <_1 L$  and  $K \in S$ . Then  $LK <_r K$ , so  $LK \in S$ . Since  $LK <_1 A$  and  $A \in \mathcal{A}$ , we have that  $LK = 0$ . Thus  $K \subseteq \beta(L) \subseteq S(L)$ . This implies that  $L/S(L) \in \mathcal{A}$ .  $\square$

**Corollary 4.2.** *If  $\mathcal{A}$  is that of Proposition 4.1, then the upper radical  $u_{\mathcal{A}} = \{A \mid \text{every non-zero homomorphic image of } A \text{ contains a non-zero left } S\text{-ideal}\}$  determined by  $\mathcal{A}$  is an  $N$ -radical.*

**Proof.** Clearly,  $\beta \subseteq u_{\mathcal{A}}$ . By Proposition 4.1,  $\mathcal{A}$  is a left regular class (i.e. every non-zero left ideal of a ring in  $\mathcal{A}$  can be homomorphically mapped onto a non-zero ring in  $\mathcal{A}$ ). Hence by [5, Theorem 9],  $u_{\mathcal{A}}$  is a left strong radical. We shall prove that it is left hereditary. For, let  $L <_1 A$  and  $A \in u_{\mathcal{A}}$ . If  $T \triangleleft L$  and  $L/T \in \mathcal{A}$ , then for an ideal  $I$  of  $A$  maximal with respect to  $I \cap L \subseteq T$  we have by Proposition 4.1, that  $A/I \in \mathcal{A}$ . However  $A \in u_{\mathcal{A}}$ , so  $A = I$ . Consequently,  $T = L$  and we are done.  $\square$

In particular we have:

**Corollary 4.3.**  $\overline{\text{Nil}} = \{A \mid \text{every non-zero homomorphic image of } A \text{ contains a non-zero left nil ideal}\}$  is an  $N$ -radical containing the class of all nil rings.

**Corollary 4.4.** *If  $R$  is a simple ring,  $R_2$  is a nil ring and (a)  $R_1$  is a  $\beta$ -radical ring with  $l(R_1) + r(R_1) \neq 0$  (in particular, if  $R_1$  is an  $M$ -nilpotent ring) or (b)  $R_1$  is a nil PI ring, then there is a simple nil ring.*

**Proof.** By Theorem 1.8 and Theorem 1.10,  $R \in \overline{\text{Nil}}$ . Hence,  $R$  contains a non-zero left nil ideal  $L$ . Since  $R \notin \beta$ ,  $RL/\beta(RL)$  is a non-zero nil ring. We claim that  $RL/\beta(RL)$  is a simple ring. For, if  $\beta(RL) \subseteq I \triangleleft RL$ , then  $RLIR \triangleleft R$ . Thus  $RLIR = 0$  or  $RLIR = R$ . In the former case  $RLI = 0$ , so  $I = \beta(RL)$ . If  $RLIR = R$ , then  $I \supseteq RLIRL = RL$ , so  $I = RL$ . This proves the claim and the corollary.  $\square$

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